AUTOMATIC CONTINUITY IN METRIC STRUCTURES

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Contents

1. Introduction
2. First order metric structures
3. Examples

Original formulation of the problem I

Problem 1.1 (A.L. Cauchy)

What are the functions $\pi : \mathbb{R} \to \mathbb{R}$ *satisfying the functional equation*

$$\pi(x+y) = \pi(x) + \pi(y)? \tag{1}$$

Or, are all the functions $\pi : (\mathbb{R}, +) \to (\mathbb{R}, +)$ *satisfying such equation continuous?*

Original formulation of the problem II

Well, it depends on the (in)famous Axiom of Choice.

Theorem 1.2 (H. Steinhaus)

Let A be a set of positive measure in $\mathbb{R}$, then the difference set

$$A - A = \{a - b \mid a, b \in A\}$$

contains an open neighborhood of zero.

It can be shown:

1. If π is measurable and satisfy (1), then it is continuous.
2. **However** the solution for Cauchy functional equation problem depends on the Axiom of Choice.

Original formulation of the problem III

3. In ZFC, there exist discontinuous homomorphisms between $(\mathbb{R}, +)$ and $(\mathbb{R}, +)$, while there are models of ZF where every set of $\mathbb{R}$ is measurable and then every homomorphism is continuous.

Alternative approach to the problem I

Definition 1.3

Let $(G, \cdot, ^{-1})$ be a group.

1. G is a Polish group if it is a topological group, there is a compatible complete metric with the topology of G, and it is separable.
2. A Polish group is said to satisfy the *automatic continuity property* if every homomorphism into a Polish group is continuous.
3. G is said to be *k-Steinhaus*, if there is $k \geq 1$ such that for every $W \subset G$ covering G by countably many right translates, W^k contains a neighborhood of the identity in G.

Alternative approach to the problem II

Remark 1

For Polish groups, Rosendal [Ros19] narrowed the answer to the **Cauchy functional equation problem** to a quadrichotomy, depending on the use of the Axiom of Choice, provided whether the *characteristic group of* π:

$$N = \bigcap_{V, V \text{is a nbd of } 1_G} \overline{\pi[V]},$$

1. is trivial,
2. compact and connected,
3. compact,
4. non of the previous.

Alternative approach to the problem III

Problem 1.4

Let $(G, \cdot)$ be a Polish group. When does G satisfy the automatic continuity property?

If the group is k-Steinhaus for some k, then it satisfies the automatic continuity property [RS07].

Languages and structures I

Example 2.1 (Languages and structures)

Let's consider the structure $\mathcal{R} = (\mathbb{R}, +, \leq, 0, 1)$, here the language is $L = \{+, \leq, 0, 1\}$ consisting of a binary function symbol, a binary relation symbol, and two constant symbols. All this logical symbols are interpreted in $\mathcal{R}$ as we expect.

Languages and structures II

Definition 2.2 (First order metric structures)

Consider the countable first-order relational language

$$L_{\mathsf{dist}} = \{D_r \mid r \in \mathbb{Q}_+\},$$

where each D_r is a binary relational symbol and $\mathbb{Q}_+ = \mathbb{Q} \cap [0, \infty[$. An L_{dist}-structure $\mathcal{X} = \langle X, \{D_r^{\mathcal{X}}\}_{r \in \mathbb{Q}_+} \rangle$ is said to be a *metric* L_{dist}-*structure* provided that there is a metric d on X so that

$$\mathcal{X} \models D_r(x, y) \Leftrightarrow d(x, y) \leqslant r.$$

Remark 2

More generally, if L is a first-order language containing L_{dist}, then an L-structure $\mathcal{X}$ will be said to be a *metric L-structure* provided that the following conditions hold.

1. for every k-ary function symbol $F \in L$, the interpretation

$$F^{\mathcal{X}} \colon X^k \to X$$

   is continuous with respect to the metric topology on X,

2. for every k-ary relation symbol $R \in L$, the interpretation

$$R^{\mathcal{X}} = \{(x_1, \ldots, x_k) \in X^k \mid \mathcal{X} \models R(x_1, \ldots, x_k)\}$$

   is closed with respect to the metric topology on X. A metric L-structure $\mathcal{X}$ will be said to be *separable* or *complete* provided that the universe X of $\mathcal{X}$ is respectively separable or complete with respect to the associated metric d.

Definition 2.3

Let $\mathcal{X}$ be a first-order metric structure,

1. an *automorphism of* $\mathcal{X}$ is a bijection $g : X \to X$ that preserves every symbol in L.
2. note that under the composition of functions, $\mathrm{Aut}(\mathcal{X})$ becomes a group.
3. When X is a uncountable separable complete metric space, $\mathrm{Aut}(\mathcal{X})$ will henceforth be given its topology of pointwise convergence on (X, d).

Example 2.4

When $L = L_{\mathrm{dist}}$ an automorphism is just a surjective isometry of X, $\mathrm{Isom}(\mathrm{X}, \mathrm{d})$, as they only need to preserve the distance predicate symbols D_r

Note that the metric d on X induces a compatible metric d_∞ on each X^k by the formula

$$d_\infty(\overline{a}, \overline{b}) = \max_i d(a_i, b_i)$$

Definition 2.5

For $\overline{a}$ a finite tuple in $\mathcal{X}$ and $\varepsilon > 0$:

1. Define
$$N(\overline{a}, \varepsilon) = \{g \in \mathrm{Aut}(\mathcal{X}) \mid d_\infty(g\overline{a}, \overline{a}) < \varepsilon\}$$
and observe that these sets are a neighbourhood basis at the identity of $\mathrm{Aut}(\mathcal{X})$.
2. The the orbit of the tuple $\overline{a}$ under the action of $\mathrm{Aut}(\mathcal{X})$ is the set
$$\mathcal{O}(\overline{a}) = \left\{g\overline{a} \mid g \in \mathrm{Aut}(\mathcal{X})\right\}$$

Definition 2.6

1. For a tuple $\overline{a}$, $\mathrm{Aut}(\mathcal{X}, \overline{a})$ is the pointwise stabilizer of $\overline{a}$, i.e, the automorphisims of $\mathcal{X}$ that fixes a,

$$\mathrm{Aut}(\mathcal{X}, \overline{a}) = \{g \in \mathrm{Aut}(\mathcal{X}) \mid g\overline{a} = \overline{a}\}$$

2. A family $\mathcal{B} \subseteq \bigcup_{k \geqslant 1} X^k$ is said to be a *basis* for $\mathrm{Aut}(\mathcal{X})$ provided that, for all tuples $\overline{a}$ and $\varepsilon > 0$, there are $\overline{b} \in \mathcal{B}$ and $\eta > 0$ so that

$$\mathrm{Aut}(\mathcal{X}, \overline{b}) \subseteq \mathrm{Aut}(\mathcal{X}, \overline{a}) \quad \text{and} \quad N(\overline{b}, \eta) \subseteq N(\overline{a}, \varepsilon).$$

3. $$\mathcal{O}(\overline{b}/\overline{a}) = \left\{\overline{c} \mid \mathcal{O}(\overline{c}, \overline{a}) = \mathcal{O}(\overline{b}, \overline{a})\right\} = \mathrm{Aut}(\mathcal{X}, \overline{a}) \cdot \overline{b}.$$

Hrushovski and Extension Properties I

Definition 2.7

A first-order structure $\mathcal{X}$ is said to have the *Hrushovski property* provided that, for any finite collection $\{\phi_i\}_{i\in I}$ of isomorphisms

$$\mathcal{A}_i \xrightarrow{\phi_i} \mathcal{B}_i$$

between finitely generated substructures $\mathcal{A}_i, \mathcal{B}_i \subseteq \mathcal{X}$, there is a finitely generated substructure $\mathcal{D} \subseteq \mathcal{X}$ containing all the $\mathcal{A}_i$ and automorphisms $f_i \in \mathrm{Aut}(\mathcal{X})$ so that each f_i extends ϕ_i and leaves $\mathcal{D}$ invariant.

Hrushovski and Extension Properties II

Definition 2.8

Assume also that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are substructures of a first-order structure $\mathcal{X}$ with $\mathcal{A} \subseteq \mathcal{B} \cap \mathcal{C}$ and let $\mathcal{D}$ denote the substructure of $\mathcal{X}$ generated by $\mathcal{B} \cup \mathcal{C}$. We say that $\mathcal{B}$ *is independent from* $\mathcal{C}$ *over* $\mathcal{A}$, written

$$\mathcal{B} \underset{\mathcal{A}}{\perp} \mathcal{C},$$

provided that whenever ϕ and ψ are automorphisms of respectively $\mathcal{B}$ and $\mathcal{C}$, both leaving $\mathcal{A}$ invariant and so that $\phi|_{\mathcal{A}} = \psi|_{\mathcal{A}}$, then there is an automorphism σ of $\mathcal{D}$ extending both ϕ and ψ.

Hrushovski and Extension Properties III

Definition 2.9

A first-order structure $\mathcal{X}$ is said to have the *extension property* provided that, for all finitely generated substructures $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{X}$ satisfying $\mathcal{A} \subseteq \mathcal{B} \cap \mathcal{C}$, there is some $g \in \mathrm{Aut}(\mathcal{X})$ so that $g|_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}}$ and

$$\mathcal{B} \underset{\mathcal{A}}{\downarrow} g[\mathcal{C}].$$

Main result

Theorem 2.10 (C. Rosendal, L. Suarez)

Let $\mathcal{X}$ be a separable complete metric L-structure in a countable language $L \supseteq L_{\mathrm{dist}}$ and assume that $\mathcal{X}$ has the extension and Hrushovski properties. Fix also a basis $\mathcal{B}$ for $\mathrm{Aut}(\mathcal{X})$ and suppose also that, for all $\bar{a} \in \mathcal{B}$ and $\varepsilon > 0$,

$$\mathrm{int}\big\{(g, f) \in N(\bar{a}, \varepsilon) \times \mathrm{Aut}(\mathcal{X}) \;\big|\; \mathcal{O}(\bar{a}/g\bar{a}) \cap \mathcal{O}(f\bar{a}/\bar{a}) \neq \emptyset\big\} \neq \emptyset.$$

Then $\mathrm{Aut}(\mathcal{X})$ has the automatic continuity property.

Here we were able to show that $\mathrm{Aut}(\mathcal{X})$ is k-Steinhaus with (interesting fact) $k = 64$.

The Urysohn space I

The *Urysohn metric space* $\mathbb{U}$ first constructed by P. S. Urysohn is a separable complete metric space satisfying the following *metric extension property*.

For any finite metric space X, any subspace $Y \subseteq X$ and any isometric embedding

$$Y \xrightarrow{\phi} \mathbb{U},$$

there exists an extension $\tilde{\phi}$ of ϕ to an isometric embedding

$$X \xrightarrow{\tilde{\phi}} \mathbb{U}.$$

As shown by Urysohn, separability, completeness and the extension property completely determine $\mathbb{U}$ up to isometry and also imply that $\mathbb{U}$ is universal for all separable metric spaces, i.e., contains an

The Urysohn space II

isometric copy of every separable metric space. The Urysohn space $\mathbb{U}$ can be seen as a complete metric structure in the language L_{dist} and note that its automorphism group is nothing but the group $\mathrm{Isom}(\mathbb{U})$ of all isometries of $\mathbb{U}$ equipped with the topology of pointwise convergence on $\mathbb{U}$. M. Sabok [Sab19] showed that $\mathrm{Iso}(\mathbb{U})$ satisfies the automatic continuity property. However, using Theorem 2.10 we were able to provide a simpler proof of that fact.

The measure algebra I

Let (X, Σ, μ) be a standard probability space

1. We will define Meas_μ as the quotient of the Boolean algebra Σ by the ideal of null sets.
2. Meas_μ becomes a complete metric space under the metric $d\big([A],[B]\big) = \mu(A \triangle B)$.
3. Observe that Observe also that the Boolean operations $\vee$, $\wedge$ and $\neg$ are continuous functions on the metric space Meas_μ and hence the latter can be seen as a separable complete metric Boolean algebra structure with constants 0 and 1.

The measure algebra II

In particular, the automorphism group $\mathrm{Aut}(\mathrm{Meas}_\mu)$ is Polish in the topology of pointwise convergence with respect to the metric d. I. Ben Yaacov, A. Berenstein, and J. Melleray [BYBM13] showed that $\mathrm{Aut}(\mathrm{Meas}_\mu)$ satisfies the automatic continuity property. Again, by using Corollary 2.10 we were able to provide a simplier proof of the fact, only relying in the measure theoretic properties of measurable sets, seen as atoms in the Boolean algebra $\mathrm{Aut}(\mathrm{Meas}_\mu)$.

The Hilbert space

Problem 3.1 (Not easy!)

Let H be a separable infinite dimensional Hilbert space and let $U(H)$ be its unitary group, meaning,

$$U(H) = \{u \in B(H) : uu^* = u^*u = I\}.$$

with its strong operator topology, i.e, pointwise convergence topology in $U(H)$. Does $U(H)$ satisfy the automatic continuity property?

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